Shift-invariant spaces from rotation-covariant functions

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Abstract

We consider shift-invariant multiresolution spaces generated by rotation-covariant functions ρ in $\mathbb{R}^2$. To construct corresponding scaling and wavelet functions, ρ has to be localized with an appropriate multiplier, such that the localized version is an element of $L^2(\mathbb{R}^2)$. We consider several classes of multipliers and show a new method to improve regularity and decay properties of the corresponding wavelets. The wavelets are complex-valued functions, which are approximately rotation-covariant and therefore behave as Wirtinger differential operators. Moreover, our class of multipliers gives a novel approach for the construction of polyharmonic B-splines with better polynomial reconstruction properties.

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1 Shift-invariance and rotation-covariance

A multiresolution analysis of $L^2(\mathbb{R}^2)$ is specified by a sequence of shift-invariant closed subspaces

$$\{0\} \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(\mathbb{R}^2)$$

satisfying the following properties:

(i) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^2) \quad \text{and} \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$

(ii) There is a dilation matrix $A \in GL(2,\mathbb{R})$ that 1.) leaves $\mathbb{Z}^2$ invariant, i.e., $A\mathbb{Z}^2 \subset \mathbb{Z}^2$, and 2.) has eigenvalues $1 < |\lambda_1| \leq |\lambda_2|$, such that

$$f \in V_j \quad \Longleftrightarrow \quad f(A^{-j}\bullet) \in V_0.$$

(iii) There is a scaling function $\varphi \in L^2(\mathbb{R}^2)$, whose integer translates form a Riesz basis of V_0:

$$V_0 = \overline{\operatorname{span}\{\varphi(\bullet - k)\,|\,k \in \mathbb{Z}\}}.$$

The aim of this paper is to construct multiresolution analyses that are generated by rotation-covariant functions. By rotation-covariance we mean that a rotation of the argument of a function $f : \mathbb{R}^2 \to \mathbb{C}$ gets translated into a rotation of the functions' value in the complex plane:

Definition 1 *Let $f : \mathbb{R}^2 \to \mathbb{C}$ be a function. Denote by $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation operator with respect to the angle $\theta \in \mathbb{R}$*

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto R_\theta x = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1}$$

The function f is called rotation-covariant if there exists a constant $N \in \mathbb{Z}$, which is independent of x and θ, such that

$$f(R_\theta x) = f(x) \cdot e^{iN\theta}.$$

Rotation-covariant atoms have the remarkable property of encoding rotations of the analyzed function in the phase only. Let ϕ be a rotation-covariant atom. Denote by $R_\theta f = f(R_\theta \bullet)$. Then

$$\langle R_{-\theta} f, \phi\rangle = \langle f, R_\theta \phi\rangle = \langle f, \phi\rangle \cdot e^{-iN\theta}.$$

As a consequence the moduli of the coefficients are not affected by rotations of f. This feature might be interesting for transform domain image processing methods that operate on the moduli of the coefficients. Rotations of the image then don't influence the operations:

$$\text{Rotation} \circ \text{Processing} \equiv \text{Processing} \circ \text{Rotation};$$

i.e., Rotations then are intertwining for all image processing operations on the moduli.

This paper is organized as follows: In Section 2, we start with the definition of a sequence of spaces $\{V_j\}_{j\in\mathbb{Z}}$ spanned by shifts of a rotation-covariant function ρ. In the next section, we give conditions for the sequence $\{V_j\}_{j\in\mathbb{Z}}$ to form a multiresolution analysis for all dilation matrices A which operate as scaled rotation. Since multiresolution analyses are defined with respect to fixed regular grids $A^j\mathbb{Z}^2$, $j \in \mathbb{Z}$, exact rotation-covariance is not possible for φ. However, the construction of approximately rotation-covariant scaling functions is possible. In Section 4, we describe several classes of admissible localizing multipliers. It turns out that there are no continuous complex ones. We define a new class of multipliers which yield higher regularity of decomposition filters and wavelets. They also improve smoothness and polynomial reproduction properties of classical polyharmonic B-splines. This, the order of approximation and the decay are considered in Sections 5–7. We close the paper with remarks on implementation and a summary.

2 Generating function

To generate an approximately rotation-covariant multiresolution analysis, we start with a perfectly rotation-covariant function ρ and consider the shift-invariant space that is generated by integer shifts of ρ. We define ρ in the sense of distributions by

$$\widehat{\rho}(\omega_1, \omega_2) := \frac{1}{(\omega_1^2 + \omega_2^2)^\alpha (\omega_1 - i\omega_2)^N} \in \mathcal{D}'(\Omega).$$

where $\Omega = \mathbb{R}^2 \backslash \{(0,0)\}$, $\alpha \in \mathbb{R}_0^+$, $N \in \mathbb{N}$, $\alpha + N > 1$. Then $\widehat{\rho}(R_\theta \omega) = e^{-iN\theta}\widehat{\rho}(\omega)$ for $\omega = (\omega_1, \omega_2)$ and thus $\rho(R_\theta x) = e^{-iN\theta}\rho(x)$ for $x = (x_1, x_2)$. I.e., ρ as well as its Fourier transform $\widehat{\rho}$ are rotation-covariant. We consider the integer shift-invariant space $V := \overline{\text{span}}\{\rho(\bullet - k) \,|\, k \in \mathbb{Z}^2\} \subset \mathcal{D}(\Omega)$ generated by integer shifts of ρ. Our aim is to generate multiresolution analyses of $L^2(\mathbb{R}^2)$. Since $\widehat{\rho} \notin L^2(\mathbb{R}^2)$, $\rho \notin L^2(\mathbb{R}^2)$ as well, since the Fourier transform is an isometric isomorphism on $L^2(\mathbb{R}^2)$. Therefore we have to "localize" ρ in order to get a scaling function φ that is an element of $L^2(\mathbb{R}^2)$.

The function $\widehat{\rho}$ has a singularity of order $2\alpha + N$ at the origin and thus is not square-integrable. Therefore, to satisfy property (iii) of a multiresolution analysis, we need to identify a function $\varphi \in L^2(\mathbb{R}^2)$, whose translates generate a integer translation invariant subspace V_0 of $L^2(\mathbb{R}^2)$:

$$V_0 = \overline{\operatorname{span}\{\varphi(\bullet - k) \,|\, k \in \mathbb{Z}^2\}}^{L^2(\mathbb{R}^2)}.$$

This is achieved by localizing the generating function ρ, i.e., by finding a sequence $\{c_k\}_{k\in\mathbb{Z}}$ such that

$$\varphi = \sum_{k\in\mathbb{Z}} c_k \rho(\bullet + k) \in L^2(\mathbb{R}^2).$$

To this end, we eliminate the singularity of $\widehat{\rho}$ at the origin by multiplying the function by an appropriate bounded $(2\pi, 2\pi)$–periodic function ν; for example, the trigonometric polynomial

$$\nu(\omega_1, \omega_2) = \left(4\left(\sin^2 \frac{\omega_1}{2} + \sin^2 \frac{\omega_2}{2}\right)\right)^{\alpha + \frac{N}{2}}.$$

Then

$$\widehat{\varphi}(\omega_1, \omega_2) = \frac{\nu(\omega_1, \omega_2)}{(\omega_1^2 + \omega_2^2)^{\alpha} \cdot (\omega_1 - i\omega_2)^N} \tag{2}$$

is bounded in $(\omega_1, \omega_2) = (0, 0)$. Moreover, this function is square-integrable for $\alpha + \frac{N}{2} > \frac{1}{2}$, and has fast decay

$$|\widehat{\varphi}(\omega_1, \omega_2)| = \mathcal{O}(\|(\omega_1, \omega_2)\|^{-2\alpha - N}) \tag{3}$$

for $\|(\omega_1, \omega_2)\| \to \infty$. This results in a function $\varphi(x_1, x_2) \in L^2(\mathbb{R}^2)$ with explicit space domain representation

$$\varphi(x) = \sum_{k\in\mathbb{Z}^2} \nu_k \rho(x + k) \quad \text{for almost all} \quad x = (x_1, x_2) \in \mathbb{R}^2. \tag{4}$$

Here, $(\nu_k)_{k\in\mathbb{Z}}$ denotes the sequence of Fourier coefficients of the $(2\pi, 2\pi)$–periodic multiplier $\nu(\omega_1, \omega_2)$ and ρ is the inverse Fourier transform of the Hadamard partie finie $Pf(\widehat{\rho})$:

Proposition 2 *For $\alpha \notin \mathbb{N}$*

$$\rho(x_1, x_2) = d_1(x_1^2 + x_2^2)^{\alpha-1}(x_1 + ix_2)^N, \tag{5}$$

and for $\alpha \in \mathbb{N}$

$$\begin{aligned}\rho(x_1, x_2) &= \\ &= d_2(x_1^2 + x_2^2)^{\alpha-1}(x_1 + ix_2)^N \left(\ln \pi\sqrt{x_1^2 + x_2^2} + d_3\right)\end{aligned} \tag{6}$$

for coefficients $d_1, d_2, d_3 \in \mathbb{C}$ depending on α and N only.

We give the proof in Section 2.2 together with explicit formulas for the coefficients.

Lemma 3 *(i) For $s < 2\alpha + N - 1$ the function φ is element of the Sobolev space*

$$\varphi \in W_2^s(\mathbb{R}^2).$$

(ii) If $\alpha + \frac{N}{2} > 1$, the function φ is bounded, uniformly continuous, and vanishes at infinity.

(iii) Let $\beta \in \mathbb{N}^2$ denote a multi-index, $\omega^\beta = \omega_1^{\beta_1}\omega_2^{\beta_2}$ the corresponding monomial, and $D^\beta = \frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\frac{\partial^{\beta_2}}{\partial x^{\beta_2}}$ the corresponding differential operator.

Then $(\bullet)^\beta \widehat{\psi} \in L^1(\mathbb{R}^2)$, as long as $2\alpha + N - |\beta| > 2$.

In this case, $D^\beta\varphi$ is bounded, uniformly continuous, and vanishes at infinity.

PROOF. (i) follows from the decay properties of $\widehat{\varphi}$. (ii) For $\alpha + \frac{N}{2} > 1$ the function $\widehat{\varphi} \in L^1(\mathbb{R}^2)$. The claim follows by the inverse Fourier transform. The same arguments yield (iii). □

Note 1 *If $\alpha + \frac{N}{2} \in \mathbb{N}$ and ν is a trigonometric polynomial, then $\varphi \in C^{2\alpha-2+N}$. In this case, ν has finitely many non-vanishing Fourier coefficients. Thus by (4), φ is a finite sum of shifted $\rho \in C^{2\alpha-2+N}$.*

2.2 Explicit representation of ρ

We now give the proof for Proposition 2.

Consider

$$\widehat{\rho}_{\alpha,N}(\omega_1, \omega_2) := \frac{1}{(\omega_1^2 + \omega_2^2)^\alpha(\omega_1 - i\omega_2)^N}.$$

Then the recurrence

$$(\omega_1 + i\omega_2)^N \widehat{\rho}_{\alpha+N,0}(\omega_1, \omega_2) = \widehat{\rho}_{\alpha,N}(\omega_1, \omega_2)$$

holds. We are seeking the inverse Fourier transform of $\widehat{\rho} = \widehat{\rho}_{\alpha,N}$. Since for $2\alpha + N \geq 2$ the function is not integrable in a neighborhood of the origin we consider the Hadamard final part $Pf(\widehat{\rho}_{\alpha,N})$ [1,2]. This can be omitted for $2\alpha + N < 2$, because it does not change the result.

The polynomial $(\omega_1 + i\omega_2)^N$ acts as a differential operator of Wirtinger type in space domain:

$$\begin{aligned}
\rho_{\alpha,N}(x_1, x_2) &= \mathcal{F}^{-1}(Pf(\widehat{\rho}_{\alpha,N}))(x_1, x_2) \\
&= \mathcal{F}^{-1}(Pf((\omega_1 + i\omega_2)^N \widehat{\rho}_{\alpha+N,0}))(x_1, x_2) \\
&= \left(-i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^N (\mathcal{F}^{-1} Pf(\widehat{\rho}_{\alpha+N,0}))(x_1, x_2) \\
&= \left(-i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^N \left(\frac{1}{(2\pi)^2}(\mathcal{F} Pf(\widehat{\rho}_{\alpha+N,0}))(-x_1, -x_2)\right). \quad (7)
\end{aligned}$$

For radial functions of variable $r \in \mathbb{R}^+$ (see e.g. [1, pp. 44 ff. and 257 ff.])

$$\begin{aligned}
\mathcal{F}(r^m) &= \frac{1}{\pi^{m+1}} \cdot \frac{\Gamma(\frac{m+2}{2})}{\Gamma(-\frac{m}{2})} r^{-(m+2)} \quad \text{for } 0 > \operatorname{Re} m > -2, \\
\mathcal{F}(Pf(r^m)) &= \frac{1}{\pi^{m+1}} \cdot \frac{\Gamma(\frac{m+2}{2})}{\Gamma(-\frac{m}{2})} r^{-(m+2)} \quad \text{for } \operatorname{Re} m < -2, \\
\mathcal{F}(r^m) &= \frac{1}{\pi^{m+1}} \frac{\Gamma(\frac{m+2}{2})}{\Gamma(-\frac{m}{2})} Pf(r^{-(m+2)}) \quad \text{for } \operatorname{Re} m > 0,
\end{aligned}$$

as long as $m \notin \mathbb{Z}$. In the case $m \in \mathbb{Z}$, further terms—logarithmic terms or differential operators—have to be added; this we shall consider later.

First we consider the case $\alpha \notin \mathbb{N}$. The equations above yield with $m = -2(\alpha + N)$

$$\begin{aligned}
&\mathcal{F}(Pf(\widehat{\rho}_{\alpha+N,0}))(x_1, x_2) = \mathcal{F} Pf\left(\frac{1}{r^{2(\alpha+N)}}\right)\Bigg|_{r=\sqrt{x_1^2+x_2^2}} \\
&= \frac{1}{\pi^{-2(\alpha+N)+1}} \cdot \frac{\Gamma\left(\frac{-2(\alpha+N)+2}{2}\right)}{\Gamma\left(-\frac{-2(\alpha+N)}{2}\right)} r^{-(-2(\alpha+N)+2)}\Bigg|_{r=\sqrt{x_1^2+x_2^2}}
\end{aligned}$$

$$= \pi^{2(\alpha+N)-1} \frac{\Gamma(-(\alpha+N)+1)}{\Gamma(\alpha+N)} (x_1^2+x_2^2)^{\alpha+N-1}. \tag{8}$$

In space domain we consider $u_{\alpha,N}(x_1,x_2) := (x_1^2+x_2^2)^{\alpha}(x_1+ix_2)^N$. This function fulfills the recursion formula

$$\left(-i\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right) u_{\alpha,N} = (-2i\alpha) u_{\alpha-1,N+1}.$$

For $k \in \mathbb{N}$,

$$\left(-i\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)^k u_{\alpha+N-1,0} = (-2i)^k \frac{\Gamma(\alpha+N-1)}{\Gamma(\alpha+N-1-k)} u_{\alpha+N-1-k,k}. \tag{9}$$

Using (7), (8) and (9) we get

$$\rho(x_1,x_2) = \rho_{\alpha,N}(x_1,x_2) = d_1(\alpha,N)(x_1^2+x_2^2)^{\alpha-1}(x_1+ix_2)^N$$

with

$$d_1(\alpha,N) = \frac{\pi^{2(\alpha+N-1)-1}}{4}(-2i)^N \frac{\Gamma(-(\alpha+N)+1)}{\Gamma(\alpha+N)} \cdot \frac{\Gamma(\alpha+N-1)}{\Gamma(\alpha-1)}. \tag{10}$$

This gives (5).

Now, we come to the second case $\alpha \in \mathbb{N}$. For $h \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ [1],

$$\begin{aligned} \mathcal{F}\left(Pf\left(\frac{1}{r^{2+2h}}\right)\right) = \frac{\pi^{1+2h}}{\Gamma(1+h)} \cdot 2\frac{(-1)^h}{h!} r^{2h} \times \\ \left(\ln\frac{1}{\pi r} + \frac{1}{2}\left(1+\frac{1}{2}+\ldots+\frac{1}{h}-\gamma\right) + \frac{1}{2}\frac{\Gamma'(1+h)}{\Gamma(1+h)}\right). \end{aligned}$$

Here, $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \ln n\right) = 0.5772\ldots$ is the Euler–Mascheroni constant. For $h=0$ the sum $1+\frac{1}{2}+\ldots+\frac{1}{h}$ must be replaced by zero. Hence, for $\alpha \in \mathbb{N}$ and $N \in \mathbb{N}$, we have with (7) and $h = \alpha+N-1$

$$\begin{aligned} &\rho_{\alpha,N}(x_1,x_2) \quad = \\ &= \mathcal{F}^{-1}(Pf(\widehat{\rho}_{\alpha,N}))(x_1,x_2) \\ &= \left(-i\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)^N \left(\frac{1}{(2\pi)^2}\mathcal{F}Pf(\widehat{\rho}_{\alpha+N,0})(-x_1,-x_2)\right) \end{aligned}$$

$$= \left(-i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^N \left(\frac{1}{(2\pi)^2}\,\mathcal{F}Pf\left(\frac{1}{r^{2(\alpha+N)}}\right)\Big|_{r=\sqrt{x_1^2+x_2^2}}\right)$$

$$= \left(-i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^N \Bigg(c_1(\alpha,N)(x_1^2+x_2^2)^{\alpha+N-1} \times$$

$$\left[\ln\frac{1}{\pi\sqrt{x_1^2+x_2^2}} + c_2(\alpha,N)\right]\Bigg),$$

where

$$c_1(\alpha,N) = \frac{1}{(2\pi)^2}\frac{\pi^{2(\alpha+N-1)+1}}{\Gamma(\alpha+N)}2\frac{(-1)^{\alpha+N-1}}{(\alpha+N-1)!}$$

and

$$c_2(\alpha,N) = \frac{1}{2}\left(1+\frac{1}{2}+\ldots+\frac{1}{\alpha+N-1}-\gamma\right)+\frac{1}{2}\frac{\Gamma'(\alpha+N)}{\Gamma(\alpha+N)}.$$

Then the Leibniz formula yields

$$\rho_{\alpha,N}(x_1,x_2) \quad =$$

$$= c_1(\alpha,N)c_2(\alpha,N)\left(-i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^N u_{\alpha+N-1}(x_1,x_2)$$

$$-\ c_1(\alpha,N)\left(-i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^N \left(u_{\alpha+N-1,0}(x_1,x_2)\ln\left(\pi\sqrt{x_1^2+x_2^2}\right)\right)$$

$$= c_1(\alpha,N)c_2(\alpha,N)(-2i)^N\frac{\Gamma(\alpha+N-1)}{\Gamma(\alpha-1)}u_{\alpha-1,N}(x_1,x_2)$$

$$-\ c_1(\alpha,N)\sum_{k=0}^{N}\binom{N}{k}(-2i)^k\frac{\Gamma(\alpha+N-1)}{\Gamma(\alpha+N-1-k)}u_{\alpha+N-1-k,k}(x_1,x_2)\times$$

$$\left(-i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^{N-k}\ln\left(\pi\sqrt{x_1^2+x_2^2}\right).$$

For $k \in \mathbb{N}$,

$$\left(-i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^k \ln\left(\pi\sqrt{x_1^2+x_2^2}\right) = \beta(k)\frac{1}{(x_1-ix_2)^k},$$

with $\beta(1) = -i$ and $\beta(k+1) = 2ik\beta(k) = (2i)^k k!(-i)$. This can be seen by induction.

Thus,

$$
\begin{aligned}
\rho_{\alpha,N}(x_1,x_2) \quad &= \\
= c_3(\alpha,N)u_{\alpha-1,N}(x_1,x_2)\ln\left(\pi\sqrt{x_1^2+x_2^2}\right) &+ c_4(\alpha,N)u_{\alpha-1,N}(x_1,x_2),
\end{aligned}
$$

where

$$
c_3(\alpha,N) = -c_1(\alpha,N)(-2i)^N\frac{\Gamma(\alpha+N-1)}{\Gamma(\alpha-1)}
$$

and

$$
\begin{aligned}
c_4(\alpha,N) = c_1(\alpha,N)\Gamma(\alpha+N-1)(-2i)^N \times \\
\left(\frac{c_2(\alpha,N)}{\Gamma(\alpha-1)} - \sum_{k=0}^{N-1}\binom{N}{k}(-1)^k\frac{(N-k)!(-i)}{\Gamma(\alpha+N-1-k)}\right).
\end{aligned}
$$

This gives (6) with $d_2(\alpha,N) = c_3(\alpha,N)$ and $d_3(\alpha,N) = c_4(\alpha,N)/c_3(\alpha,N)$.

3 Multiresolution analyses

Now we show that the function φ constructed in (2) is a valid scaling function, and that it generates a multiresolution analysis of $L^2(\mathbb{R}^2)$ for a set of dilation matrices. We state the conditions for φ to be a scaling function, and specify the corresponding multipliers ν.

The function φ in (4) satisfies a refinement relation

$$
\varphi(A^{-1}\bullet) = \sum_{k\in\mathbb{Z}^2} h_k\varphi(\bullet - k) \quad \text{in } L^2(\mathbb{R}^2) \tag{11}
$$

for a set of dilation matrices A. Consider the scaled rotation matrices $A \in GL(2,\mathbb{R})$ of the form

$$
A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \sqrt{|\det A|}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \sqrt{a^2+b^2}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{12}
$$

with $a,b\in\mathbb{Z}$, $\theta = \arccos\frac{a}{\sqrt{a^2+b^2}}$, and eigenvalues $1 < |\lambda_1| = |\lambda_2|$. This includes the quincunx case $a = b = 1$. Relation (11) is equivalent to the existence of a $2\pi\mathbb{Z}^2$–periodic function $H \in L^2(\mathbb{T}^2)$ of the form

$$
H(e^{i\omega}) = |\det A|\frac{\widehat{\varphi}(A^T\omega)}{\widehat{\varphi}(\omega)} \quad = \quad |\det A|\frac{\widehat{\rho}(A^T\omega)\nu(A^T\omega)}{\widehat{\rho}(\omega)\nu(\omega)}
$$

$$= \frac{\nu(A^T\omega)}{\nu(\omega)} \cdot \frac{1}{(a^2+b^2)^{\alpha-1}(a-ib)^N}, \quad \omega \in \mathbb{R}^2. \tag{13}$$

The function H is a refinement filter with Fourier coefficients h_k defined in (11). We can formulate for general multipliers ν:

Theorem 4 *Let $\alpha \in \mathbb{R}^+$ and $N \in \mathbb{N}$ be fixed. Let the multiplier $\nu(\omega_1, \omega_2)$ be a bounded, $2\pi\mathbb{Z}^2$-periodic function in $\mathbb{R}^2$ that fulfills the following properties:*

(i) $\left|\frac{\nu(\omega_1,\omega_2)}{(\omega_1^2+\omega_2^2)^\alpha(\omega_1 - i\omega_2)^N}\right| \to c$ *for* $\|(\omega_1,\omega_2)\| \to 0$, *and a positive constant* c.

(ii) $\nu(\omega_1,\omega_2) \neq 0$ *for all* $(\omega_1,\omega_2) \in [-\pi,\pi]^2 \setminus \{(0,0)\}$.

Then $\widehat{\varphi} = \nu \cdot \widehat{\rho}$ is the Fourier transform of a scaling function φ which generates a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^2)$ with dilation matrix A:

$$V_j = \overline{\operatorname{span}\{|\det A|^{j/2}\varphi(A^j \bullet - k), k \in \mathbb{Z}^2\}}.$$

Note 2 *This construction of localizing homogeneous polynomials $\rho^{-1}(\omega) = p(\omega) = \sum_{|\beta|=m} a_\beta \omega^\beta$, $\omega \in \mathbb{R}^2$, β multi-index with $|\beta| = m > 2$, is similar to the one for real-valued polyharmonic B-splines in [3]. However, condition (i) in our Theorem 4 is weeker than the respective condition in [3, (1.17)]. In fact, there it is supposed that $\nu(\omega) - \rho^{-1}(\omega) = \mathcal{O}(\|\omega\|_\infty^{m+1+n})$ for $\omega \to \infty$ and some $n \in \mathbb{N}_0$. In our case, for real-valued multipliers ν and $N \geq 1$, this condition is never met. Nonetheless, we generate valid multiresolution analyses.*

Before we prove Theorem 4, we show that the translates of φ form a Riesz sequence:

Proposition 5 *Under the assumptions of Theorem 4, the autocorrelation function*

$$M(\omega) := \sum_{k\in\mathbb{Z}^2} |\widehat{\varphi}(\omega + 2\pi k)|^2 \tag{14}$$

is bounded by

$$\begin{aligned} 0 < c &\leq \sum_{k\in\mathbb{Z}^2} |\widehat{\varphi}(\omega + 2\pi k)|^2 \\ &\leq C + B\left(\frac{2\pi}{4\alpha+2N-2} + 8\zeta(4\alpha+2N) + 3\cdot 4^{2\alpha+N}\right), \end{aligned} \tag{15}$$

where c is the minimum of $\left|\frac{\nu(\omega_1,\omega_2)}{(\omega_1^2+\omega_2^2)^\alpha(\omega_1-i\omega_2)^N}\right|$, C the maximum of the same term, and B the maximum of $|\nu|$ in $[-\pi,\pi]^2$, respectively. The function ζ denotes the Riemann zeta function.

PROOF. We show that $\{\varphi(\bullet - k), k \in \mathbb{Z}\}$ is a Riesz sequence by bounding the autocorrelation function. First, we estimate the function $\sum_{k\in\mathbb{Z}^2} |\widehat{\varphi}(\omega + 2\pi k)|^2$ almost everywhere from above. Because of its $(2\pi, 2\pi)$–periodicity, it is sufficient to consider $\omega \in [\,-\pi, \pi]^2$ only.

$$\begin{aligned}\sum_{k\in\mathbb{Z}^2} |\widehat{\varphi}(\omega + 2\pi k)|^2 \quad &= \\ = \left|\frac{\nu(\omega_1, \omega_2)}{(\omega_1^2 + \omega_2^2)^\alpha(\omega_1 - i\omega_2)^N}\right|^2 &+ \sum_{\substack{k\in\mathbb{Z}^2 \\ k\neq(0,0)}} \frac{|\nu(\omega_1, \omega_2)|^2}{|(\omega_1 + 2\pi k_1)^2 + (\omega_2 + 2\pi k_2)^2|^{\alpha+\frac{N}{2}}}\end{aligned}$$

By (i), the first term is bounded for all $\omega \in [\,-\pi, \pi\,]^2$, since the singularity of the denominator is cancelled by ν:

$$\left|\frac{\nu(\omega_1, \omega_2)}{(\omega_1^2 + \omega_2^2)^\alpha(\omega_1 - i\omega_2)^N}\right|^2 < C$$

for some bound $C \in \mathbb{R}^+$. Since the trigonometric polynomial ν is bounded, there is a positive constant $B \geq |\nu(\omega_1, \omega_2)|^2$ such that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^2} |\widehat{\varphi}(\omega + 2\pi k)|^2 \quad &= \\ \leq C + B \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} &\left|\frac{1}{(\omega_1 + 2\pi k_1)^2 + (\omega_2 + 2\pi k_2)^2}\right|^{2\alpha+N} \qquad (16) \\ \leq C + B &\left(\frac{2\pi}{4\alpha + 2N - 2} + 8\zeta(4\alpha + 2N) + 3 \cdot 4^{2\alpha+N}\right). \qquad (17)\end{aligned}$$

This can be deduced using the fact that the sum in (16) is invariant with respect to the transforms $(\omega_1, \omega_2) \mapsto (\pm\omega_1, \pm\omega_2)$. Thus it is sufficient to consider this function for $\omega \in [\,0, \pi\,]^2$ only. Splitting the sum in (16) into sums over each of the four quadrants, and considering the axes $k_1 = 0$ and $k_2 = 0$ separately, yields (17). The sum $\sum_{k_1>0} \frac{1}{k_1^{4\alpha+N}} = \zeta(4\alpha + N)$, where $\zeta(x) = \sum_{n\geq1} \frac{1}{n^x}$ denotes Riemann's zeta-function, which exists for $x > 1$, and converges to 1 for $x \to \infty$. The sum $\sum_{k_1>0,k_2>0} \frac{1}{(k_1^2+k_2^2)^{2\alpha+N}}$ is a lower Riemann sum for the integral

$$\begin{aligned}\int_1^\infty\!\!\int_1^\infty \frac{1}{(x^2 + y^2)^{2\alpha+N}}\,dx\,dy \quad &\leq \\ \leq \int_1^\infty\!\!\int_0^{\frac{\pi}{2}} \frac{1}{(r^2)^{2\alpha+N}} r\,d\varphi\,dr \quad = \quad \frac{\pi}{2}\int_1^\infty r^{-4\alpha-2N+1}\,dr \quad &= \quad \frac{\pi}{2}\frac{1}{4\alpha + 2N - 2}.\end{aligned}$$

The autocorrelation function is strictly positive:

$$\sum_{k\in\mathbb{Z}^2} |\widehat{\varphi}(\omega+2\pi k)|^2 \quad \geq \quad \left|\frac{\nu(\omega_1,\omega_2)}{(\omega_1^2+\omega_2^2)^\alpha(\omega_1-i\omega_2)^N}\right|^2 \quad > \quad 0,$$

since at the point $(0,0)$ the value is positive by (i) and there are no other zeros of ν in $[-\pi,\pi]^2$ by (ii). $\square$

PROOF of Theorem 4. It only remains to show that the union of the spaces $V_j = \text{span}\{\varphi(A^j \bullet - k), k \in \mathbb{Z}\}$, $j \in \mathbb{Z}$, is dense in $L^2(\mathbb{R}^2)$ for all dilation matrices A of the form (12). By (15) the translates $\{\varphi(\bullet - k), k \in \mathbb{Z}^2\}$ form a Riesz sequence in $L^2(\mathbb{R}^2)$. Thus the function φ can be orthonormalized with the autocorrelation function:

$$\widehat{\Phi} = \frac{\widehat{\varphi}}{\sqrt{M}}.$$

Then the integer shifts of Φ form an orthonormal basis of V_0. Moreover, $|\widehat{\Phi}(0,0)| = 1$ and $|\widehat{\Phi}|$ is continuous in $\mathbb{R}^2$. Thus, the Riesz basis condition together with a density argument gives the result following the method of proof in [4]. Specifically, we denote by $P_j : L^2(\mathbb{R}^2) \to V_j$ the orthonormal projection. Let

$$g \in G := \{f \in L^2(\mathbb{R}^2),\ \widehat{f} \in C^\infty(\mathbb{R}^2),\ \widehat{f} \text{ has compact support}\}.$$

G is dense in $L^2(\mathbb{R}^2)$. There exists $n \in \mathbb{N}$ such that $\text{supp}\,\widehat{g} \in [-2^n\pi, 2^n\pi]$. Using Parseval's equality, we deduce

$$\begin{aligned} \|g - P_j g\|^2 &= \|g\|^2 - \sum_{k\in\mathbb{Z}^2} \left|\langle g, |\det A|^{\frac{j}{2}} \Phi(A^j \bullet - k)\rangle\right|^2 \\ &= \|g\|^2 - \frac{1}{(2\pi)^2}\|\widehat{g}\overline{\widehat{\Phi}}(A^{-j}\bullet)\|^2 \to 0 \quad \text{for } j \to \infty. \end{aligned}$$

This proves the theorem. $\square$

4 Several solutions for choosing localization filters for nearly rotation-covariant scaling functions

As mentioned before, the $2\pi\mathbb{Z}^2$–periodic function

$$\nu_1(\omega_1,\omega_2) = \left(4\left(\sin^2\left(\frac{\omega_1}{2}\right)+\sin^2\left(\frac{\omega_2}{2}\right)\right)\right)^{\alpha+\frac{N}{2}} \tag{18}$$

is a valid multiplier, since ν_1 makes the singularity of $\widehat{\rho}$ at the origin integrable. Moreover, ν_1 is non-negative, and vanishes only on the grid $2\pi\mathbb{Z}^2$. The trigonometric polynomial has regularity $\nu_1 \in C^{\lfloor 2\alpha+N\rfloor}(\mathbb{T}^2)$, and behaves as $\nu_1(\omega_1,\omega_2) = (\omega_1^2+\omega_2^2)^{\alpha+N/2} + \mathcal{O}(\|\omega\|^{2\alpha+N+1})$ in a neighborhood of the origin. The rotation-invariant terms influences the rotation-covariance of scaling functions and wavelets. To examine this more thorougly, we define:

Definition 6 *Let $f:\mathbb{R}^2\to\mathbb{C}$ be continuous. Let g be a radial function, i.e., $g(r) = a_0+a_1r+\ldots+a_mr^m$ with constant coefficients $a_0,\ldots,a_m\in\mathbb{C}$. In polar coordinates $(\omega_1,\omega_2)=(r\cos\theta,r\sin\theta)$, let*

$$f(r\cos\theta,r\sin\theta)-g(r) = R_m(r\cos\theta,r\sin\theta)r^{m+1}$$

for some bounded rest term $R_m(r\sin\theta,r\cos\theta)$ with

$$\lim_{r\to 0} R_m(r\cos\theta,r\sin\theta)\neq 0.$$

Then f is called approximately rotation-invariant of order m in a neighborhood of the origin.

Remark 7 *If f is approximately rotation-invariant of order m, then the k-th radial derivatives $\frac{\partial}{\partial r}f(r\cos\theta,r\sin\theta)$, $0\le k\le m-1$, are still approximately rotation-invariant of order $m-k$.*

Example 8*(i) The multiplier (18) is of the form $\nu_1=(\eta_1)^{\alpha+\frac{N}{2}}$ with*

$$\begin{aligned}\eta_1(\omega_1,\omega_2) &= 4\left(\sin^2\left(\frac{\omega_1}{2}\right)+\sin^2\left(\frac{\omega_2}{2}\right)\right)\\ &= 4\left(\sin^2\left(\frac{r\cos\varphi}{2}\right)+\sin^2\left(\frac{r\sin\theta}{2}\right)\right)\\ &= r^2-\frac{\cos(4\theta)+3}{48}r^4+R_5(r\sin\theta,r\cos\theta)r^6.\end{aligned} \tag{19}$$

This trigonometric polynomial is approximately rotation-invariant of order 3 in a neighborhood of the origin.

(ii) Another possible choice for such a trigonometric polynomial is

$$\begin{aligned}
\eta_2(\omega_1,\omega_2) = & \frac{8}{3}\left(\sin^2\left(\frac{\omega_1}{2}\right)+\sin^2\left(\frac{\omega_2}{2}\right)\right) \\
& +\frac{2}{3}\left(\sin^2\left(\frac{\omega_1+\omega_2}{2}\right)+\sin^2\left(\frac{\omega_1-\omega_2}{2}\right)\right) \\
= & \, r^2 - \frac{1}{12}r^4 + \frac{5-\cos 4\theta}{1440}r^6 + R_7(r\cos\theta, r\sin\theta)r^8.
\end{aligned} \tag{20}$$

This function is approximately rotation-covariant of order 5 in a neighborhood of the origin.

Multipliers of higher order of approximate rotation-invariance can be found by the same formula

$$\nu = (\eta)^{\alpha+\frac{N}{2}} \tag{21}$$

in considering the ansatz

$$\begin{aligned}
\eta(\omega_1,\omega_2) = & \sum_{n=1}^{M} a_n \left(\sin^{2n}\left(\frac{\omega_1}{2}\right)+\sin^{2n}\left(\frac{\omega_2}{2}\right)\right) \\
& +b_n\left(\sin^{2n}\left(\frac{\omega_1+\omega_2}{2}\right)+\sin^{2n}\left(\frac{\omega_1-\omega_2}{2}\right)\right) \\
= & \, r^2 + \sum_{k=2}^{m} c_k r^{2k} + R_{2m+1}(r\cos\theta, r\sin\theta)r^{2(m+1)},
\end{aligned} \tag{22}$$

for $M \in \mathbb{N}$ large enough and $(\omega_1,\omega_2) = (r\cos\theta, r\sin\theta)$ as above. Then a_n and b_n can be calculated such that the coefficients c_n do not depend on θ. This leads to a system of linear equations, which can be easily solved with e.g. Gauß elimination. Table 1 gives the coefficients for multipliers of order up to 11. They all satisfy the conditions of Theorem 4, since all $\eta(\omega_1,\omega_2) > 0$, except for $(\omega_1,\omega_2) \in 2\pi\mathbb{Z}^2$. This can be easily seen by noting $a_1 > \sum_{k=2}^{M} a_k \delta_{-1,\operatorname{sign} a_k}$, $b_1 > \sum_{k=2}^{M} b_k \delta_{-1,\operatorname{sign} b_k}$ for the negative coefficients, and the fact that $\sin^{2k}(x) \geq \sin^{2(k+1)}(x) \geq 0$ for all $k \in \mathbb{N}$, $x \in \mathbb{R}$.

With the same ansatz (22), we can construct trigonometric polynomials η, where terms r^{2k} vanish up to a certain order. Table 2 gives some examples of such multipliers with $a_1 = 1$ fixed. Obviously, all those multipliers satisfy the conditions of Theorem 4, since all coefficients are positive. Later, we shall use this special class of η to generate smooth lowpass filters and faster decaying wavelets. Moreover, they can be used to generate classical real-valued polyharmonic B-splines with higher smoothness in Fourier domain.

Multipliers of the form (21), (22) are real-valued. Thus $\widehat{\varphi}$ is bounded at the origin, but not continuous (see Figure 1). Moreover, $\varphi \notin L^1(\mathbb{R}^2)$. Nonetheless,

Order	a_1	b_1	a_2	b_2	a_3	b_3	a_4
3	4						
5	$\frac{8}{3}$	$\frac{2}{3}$					
7	$\frac{12}{5}$	$\frac{4}{5}$	$-\frac{4}{15}$				
7	$\frac{104}{45}$	$\frac{38}{45}$	$-\frac{56}{135}$	$-\frac{2}{135}$			
9	$\frac{152}{65}$	$\frac{54}{65}$	$-\frac{376}{1365}$	$\frac{6}{455}$	$-\frac{256}{4095}$		
9	$\frac{200}{87}$	$\frac{74}{87}$	$-\frac{88}{261}$	$\frac{2}{261}$	$-\frac{2048}{27405}$	$-\frac{32}{27405}$	
11	$\frac{2932}{1275}$	$\frac{1084}{1275}$	$-\frac{44}{153}$	$\frac{76}{3825}$	$-\frac{35104}{401625}$	$\frac{512}{401625}$	$-\frac{48}{2125}$

Table 1
Coefficients in ansatz (22) for multipliers of higher order of rotation-covariance.

Form of η	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4	a_5	b_5
$r^2+\mathcal{O}(r^6)$	1	$\frac{3}{2}$	$\frac{1}{3}$	$\frac{1}{2}$						
$r^2+\mathcal{O}(r^8)$	1	$\frac{3}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{8}{45}$	$\frac{4}{15}$				
$r^2+\mathcal{O}(r^{10})$	1	$\frac{3}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{8}{45}$	$\frac{4}{15}$	$\frac{4}{35}$	$\frac{6}{35}$		
$r^2+\mathcal{O}(r^{12})$	1	$\frac{3}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{8}{45}$	$\frac{4}{15}$	$\frac{4}{35}$	$\frac{6}{35}$	$\frac{128}{1575}$	$\frac{64}{525}$

Table 2
Coefficients in ansatz (22) for trigonometric polynomials η of the form $\eta(r\cos\theta, r\sin\theta) = r^2 + \mathcal{O}(r^{2m})$, $m \in \mathbb{N}$.

valid multiresolution analyses are generated. For the difference in the order of rotation-covariance and its effects on the scaling functions, compare the graphs in Figure 1 . For symmetry reasons, $\varphi(0) = 0$ for these multipliers and all $N \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$, since

$$\widehat{\varphi}(-\omega_1, \omega_2) = (-1)^N \widehat{\varphi}(\omega_1, -\omega_2) \quad \text{and} \quad \widehat{\varphi}(\omega_2, \omega_1) = (i)^N \widehat{\varphi}(\omega_1, \omega_2).$$

From Figures 1 and 2 we also see that in space domain as well as frequency domain real resp. imaginary part of φ behave as edge detectors along the real resp. imaginary axis. Since these directions are perpendicular, rotation reveals edges in every direction.

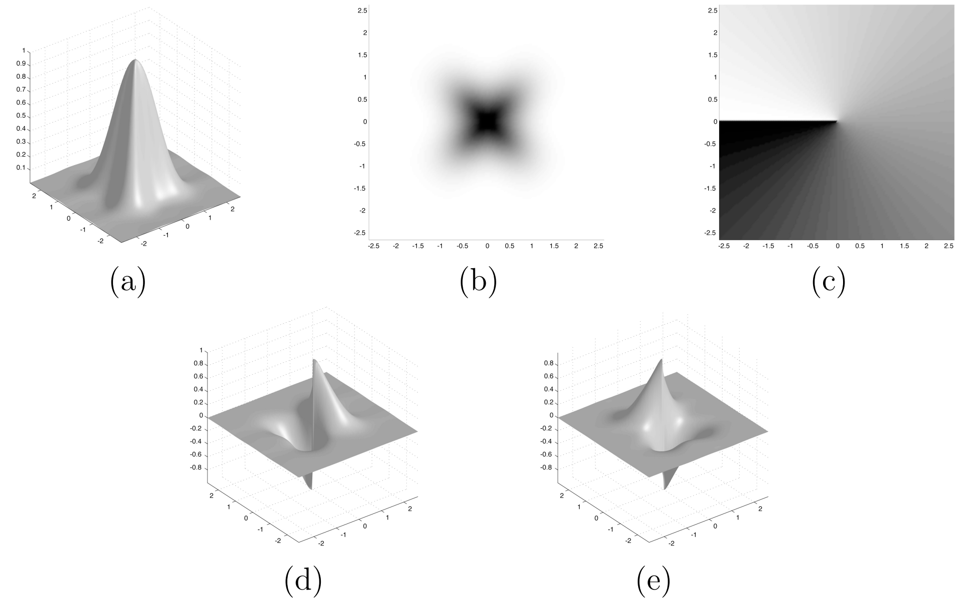
Note that polyharmonic splines correspond to the special case $N = 0$.

Proposition 9 *The class of scaling functions of the form φ as defined in Theorem 4 with multipliers of the form (21) is closed under convolution.*

Let $\varphi = \varphi_{\alpha,N,\eta}$ with $\widehat{\varphi} = (\eta)^{\alpha+N/2}\widehat{\rho}$.

(i) $\varphi_{\alpha,N,\eta} * \varphi_{\alpha_1,N_1,\eta} = \varphi_{\alpha+\alpha_1,N+N_1,\eta}$.
(ii) $\varphi_{\alpha,N,\eta} * \varphi_{\alpha,N,\eta_1} = \varphi_{\alpha,N,\eta\eta_1}$.

Localization with a multiplier of order 3



Localization with a multiplier of order 5

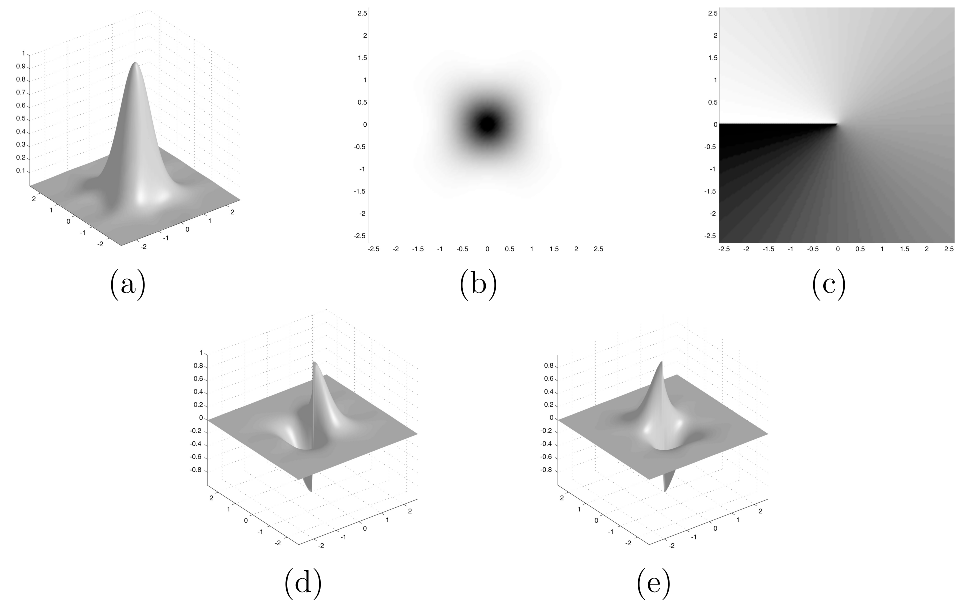
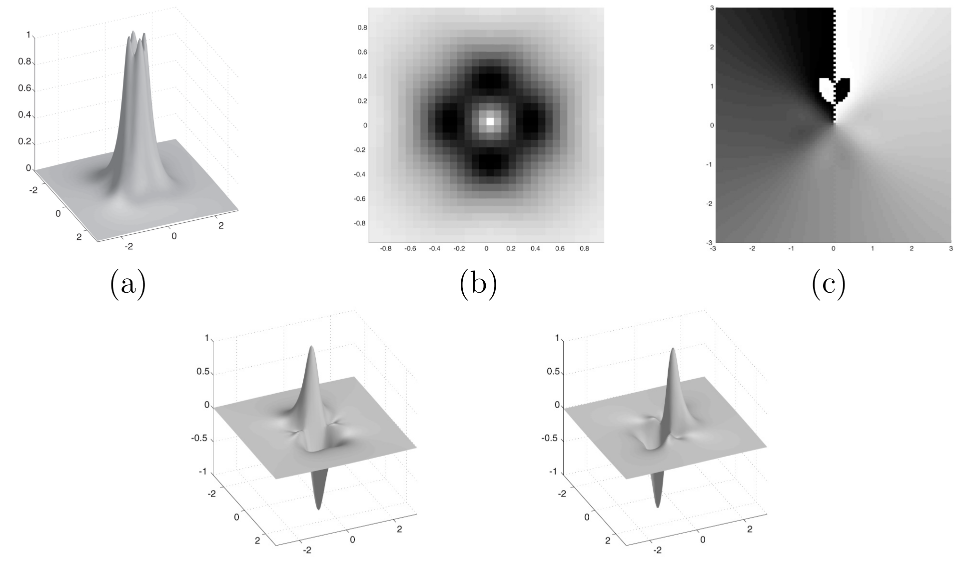


Fig. 1. The function $\widehat{\varphi} = (\eta)^{\alpha+\frac{N}{2}}\,\widehat{\rho}$ with η of different order of rotation invariance. The parameters are $\alpha = 1.5$, $N = 1$. Upper part: η as in (19) of order 3. Lower part: η as in (20) of order 5. (a) Absolute value. (b) Top view on the absolute value. Larger values are black, zeros are white. (c) Phase. The multiplier of higher order 5 yields a more isotropic $|\widehat{\varphi}|$. (d) Real and (e) imaginary part. The function is discontinuous at the origin.

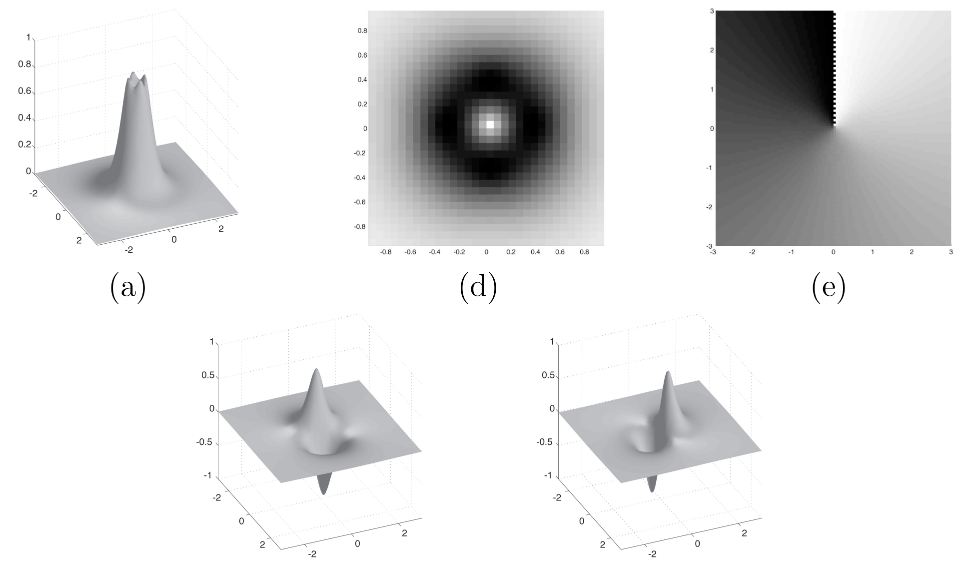
Localization with a multiplier of order 3



(a) (b) (c)

(b) (c)

Localization with a multiplier of order 5



(a) (d) (e)

(b) (c)

Fig. 2. The scaling function φ in space domain for parameters $\alpha = 1.5$ and $N = 1$. Upper part: η as in (19) of order 3. Lower part: η as in (20) of order 5. (a) Absolute value, (b) top view of the absolute value, (c) phase. (d) real and (e) imaginary part. Both functions are rotation-covariant in a neighborhood of the origin. But outside this small neighborhood, the function localized with the multiplier of order 3 clearly shows deficiencies in the homogenity of the phase, and yields a strongly non-isotropic $|\varphi|$. The multiplier of higher order 5 yields a φ, which has rotation–covariant behaviour in a much larger neighborhood of zero.

PROOF. This can be directly seen from the Fourier representation of the convolution products. $\square$

5 Decay and order of approximation

5.1 Decay

We already saw in Lemma 3 that φ is bounded, uniformly continuous and $\lim_{\|x\|\to\infty} |\varphi(x)| = 0$, as long as $\alpha + \frac{N}{2} > 1$. This is due to the fact that for these parameters α and N the function $\widehat{\varphi}$ is integrable: $\widehat{\varphi} \in L^1(\mathbb{R}^2)$. But since $\widehat{\varphi}$ is not continuous, $\varphi \notin L^1(\mathbb{R}^2)$. Thus, φ decays slower than $\frac{1}{\|x\|^2}$ for $\|x\| \to \infty$. However, since $\widehat{\varphi} \in L^2(\mathbb{R}^2)$, φ decays better $\frac{1}{\|x\|}$ as $\|x\| \to \infty$. Note that these decay rate bounds are independent of α and N. For an illustration see Figure 3.

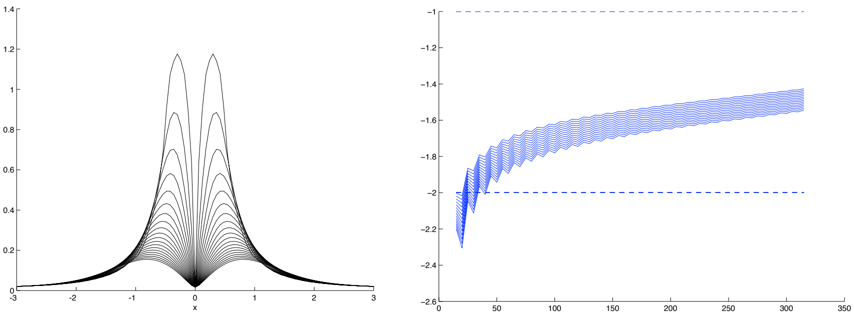


Fig. 3. The decay of the scaling function φ is slower than $1/\|x\|^2$, but faster than $1/\|x\|$. On the left, $\varphi(x, 0)$ is shown for $N = 1$ and various $\alpha \geq 1$. On the right, the branch $\ln \varphi(x, 0)/\ln(x)$ for positive x is given. The dotted lines denote the decay exponents -1 and -2.

5.2 Order of approximation

The space V_0 provides approximation order γ if, for every $f \in W_2^\gamma(\mathbb{R}^2)$,

$$E(f, V_0, h) := \min \left\{ \left\| f - s\left(\frac{\bullet}{h}\right) \right\|, s \in V_0 \right\} \leq \text{const}_{(V_0)} h^\gamma \|f\|_{W_2^\gamma(\mathbb{R}^2)}.$$

Theorem 10 *Let $\eta(\omega_1, \omega_2)$ be a trigonometric polynomial with Taylor expansion*

$$\eta(\omega_1, \omega_2) = \omega_1^2 + \omega_2^2 + \mathcal{O}(\|\omega\|^3),$$

such that $\nu = \eta^{\alpha+N/2}$ is the localizing multiplier of the scaling function $\widehat{\varphi}(\omega_1, \omega_2) = \nu(\omega_1, \omega_2)/(\omega_1^2+\omega_2^2)^\alpha/(\omega_1 + i\omega_2)^N$. Then the space V_0, which is spanned by $\mathbb{Z}^2$-shifts of φ, provides approximation order $2\alpha + N$.

PROOF. To establish the order of approximation we make use of a result by De Boor et al. [5, Theorem 1.15]. The function $1/\widehat{\varphi}$ by construction is bounded on some neighborhood of the orgin. All derivatives of $\widehat{\varphi}$ of order less or equal than $\lfloor 2\alpha + N \rfloor + 2$ are in $L^2(\mathbb{R}^2 \setminus B_\varepsilon(0,0))$ for some $\varepsilon > 0$. Since ν has zeros of order $2\alpha + N$ on the grid $2\pi\mathbb{Z}^2$ the derivatives $D^\gamma \widehat{\varphi}(\omega) = 0$ for all $|\gamma| < 2\alpha + N$ and all $\omega \in 2\pi\mathbb{Z}^2 \setminus \{(0,0)\}$. Our result then follows from the theorem of De Boor et al. $\square$

6 Multiresolution filters

6.1 Lowpass filter and convergence of infinite products

Although $\widehat{\varphi}$ has a singularity at the origin, the lowpass filter H is continuous. However, we can show for its modulus:

Proposition 11 *The modulus $\prod_{j=1}^\infty |H(A^{-j}\omega)|$ of the infinite product converges pointwise and uniformly on all compact subsets.*

PROOF. Let $\omega \in \mathbb{R}^2$ be fixed. Then $|H(0,0)| = 1$ and $||H(\omega)| - 1| < \|\omega\|^\varepsilon$ for some $\varepsilon > 0$. It is

$$\prod_{j=1}^m |H(A^{-j}\omega)| = \exp\left(\sum_{j=1}^m \ln |H(A^{-j}\omega)|\right).$$

Let j_0 be such that $||H(A^{-j}\omega)| - 1| \le \frac{1}{2}$. Then, for $j > j_0$,

$$\begin{aligned} \ln |H(A^{-j}\omega)| = \ln |1 + |H(A^{-j}\omega)| - 1| \quad &\le \quad \text{const}\, ||H(A^{-j}\omega)| - 1| \\ \le |\det A|^{-j\varepsilon} \|\omega\|^\varepsilon. & \end{aligned}$$

Thus

$$\sum_{j=1}^N \ln |H(A^{-j}\omega)| \le \sum_{j=1}^{j_0} \ln |H(A^{-j}\omega)| + \text{const} \sum_{j=j_0+1}^m |\det A|^{-j\varepsilon} \|\omega\|^\varepsilon$$

$$\leq \text{const}\left(1+\|\omega\|^{\varepsilon}\frac{1}{1-|\det A|^{-\varepsilon}}\right).$$

Taking the limit $m \to \infty$ on the left hand side proves the pointwise convergence of $\prod_{j=1}^{\infty}|H(A^{-j}\omega)|$. By the same arguments, uniform convergence on compacta is true (compare with [6, Satz 2.4.3]). $\square$

The infinite product $\prod_{j=1}^{\infty} H(A^{-J}\omega)$ does not converge pointwise due to the singularity in the origin. Since the phase of H is independent of ω, the phase of the infinite product does not converge, but periodically runs through the finite set $\{e^{ik\arg(a+ib)^N}\}_k$.

For example, for the quincunx case with $a = b = 1$ and $N = 1$, the set S has cardinality $\#S = 8$: $S = \{e^{ik\frac{\pi}{4}} : k = 0,\dots,7\}$. For $N = 2$, the set S has cardinality $\#S = 4$: $S = \{e^{ik\frac{\pi}{2}} : k = 0,\dots,3\}$.

6.2 Regularity of the lowpass filter

The smoothness of the lowpass filter H depends on the smoothness and the order of approximate rotation-covariance of the multiplier ν. With an appropriate choice of real-valued multipliers, we can adjust the smoothness of the lowpass filter H.

We assume that $\eta \in C^{\infty}$ is a real-valued multiplier, and has zeros at $2\pi\mathbb{Z}^2$. Since $\eta(A^T\omega)$ cancels the zeros of $\eta(\omega)$, the fraction $\eta(A^T\omega)/\eta(\omega)$ is continuous. Then

$$H(\omega) = \frac{1}{(a^2+b^2)^{\alpha}(a-ib)^N}\left(\frac{\eta(A^T\omega)}{\eta(\omega)}\right)^{\alpha+\frac{N}{2}}$$

is continuous.

We now consider differentiability of $\eta(A^T\omega)/\eta(\omega)$. Due to periodicity it is enough to consider the fraction in a neighborhood of the origin. Let η be of the form of ansatz (22):

$$\eta(\omega) = \eta(\omega_1,\omega_2) = r^2 + \sum_{k=2}^{m} c_k r^{2k} + R_{2m+1}(\omega) r^{2(m+1)}$$

with $r = \sqrt{\omega_1^2+\omega_2^2}$ and bounded $R_{2m+1}(\omega)$. Then

$$\eta(A^T\omega) = (a^2+b^2)\left(r^2 + \sum_{k=2}^{m} c_k (a^2+b^2)^{k-1} r^{2k} + R_{2m+1}(A^T\omega)(a^2+b^2)^m r^{2(m+1)}\right),$$

and the fraction is of the form

$$\frac{\eta(A^T\omega)}{\eta(\omega)} = (a^2+b^2)\Big(1+ \frac{\sum_{k=2}^m c_k((a^2+b^2)^{k-1}-1)r^{2k} + (R_{2m+1}(A^T\omega)(a^2+b^2)^m - R_{2m+1}(\omega))r^{2(m+1)}}{r^2+\sum_{k=2}^m c_k r^{2k} + R_{2m+1}(\omega)r^{2(m+1)}}\Big)$$
$$= (a^2+b^2)(1+r^2B(\omega))$$
$$= (a^2+b^2)(1+\mathcal{O}(\|\omega\|^2))$$

for some function $B(\omega)$ bounded in a neighborhood of the origin.

If B is even continuous, then $\eta(A^T\omega)/\eta(\omega) \in C^2$. If B is not continuous, i.e., if $m = 1$, then the sum terms are void, and $\eta(A^T\omega)/\eta(\omega) \in C^1$.

In the case $c_k = 0$ for $k = 2, \ldots, l$, we get

$$\frac{\eta(A^T\omega)}{\eta(\omega)} = (a^2+b^2)(1+\mathcal{O}(\|\omega\|^{2l})$$

in a neighborhood of the origin. Thus, multipliers of the form $\eta(\omega) = r^2 + \mathcal{O}(r^{2l})$, $l > 2$, generate smoother filters than those of the form $\eta(\omega) = r^2 + \mathcal{O}(r^4)$.

We sum up:

Theorem 12 *Let $\nu(\omega_1,\omega_2) = (\eta(\omega_1,\omega_2))^{\alpha+\frac{N}{2}}$ be a localizing multiplier, where η is according to the ansatz (22):*

$$\eta(\omega) = \eta(\omega_1,\omega_2) = r^2 + \sum_{k=2}^{m} c_k r^{2k} + R_{2m+1}(\omega) r^{2(m+1)} \tag{23}$$

with $r = \sqrt{\omega_1^2+\omega_2^2} = \|\omega\|$ and $R_{2m+1}(\omega)$ bounded.

(i) Then the lowpass filter H is C^{K_1}, where $K_1 = \min\{\lfloor\alpha+\frac{N}{2}\rfloor, 2m-1, 2\}$.
(ii) If, in addition, η is of the particular form $\eta(\omega) = r^2 + R_{2m+1}(\omega)r^{2(m+1)}$, then H is C^{K_2} with $K_2 = \min\{\lfloor\alpha+\frac{N}{2}\rfloor, 2m-1\}$.
(iii) If $\lfloor\alpha+\frac{N}{2}\rfloor \in \mathbb{N}$, then this term can be omitted from the minima K_1, K_2.

Note 3 *Multipliers η of the special form $\eta(\omega) = r^2 + \mathcal{O}(r^{2m})$, $m \geq 2$, can also be used for the localization of classical polyharmonic splines. E.g., in [7], polyharmonic B-splines in $\mathbb{R}^n$ are defined via their Fourier representation*

$$\widehat{\varphi}(\omega) = \left(\sum_{i=1}^{n} \sin^2\frac{\omega_i}{2}\right)^k / \|\frac{\omega}{2}\|^{2k} \quad \textit{for } \omega\in\mathbb{R}^n, n,k\in\mathbb{N}.$$

These functions have a representation $\widehat{\varphi}(\omega) = 1+\mathcal{O}(\|\omega\|^2)$ in a neighborhood of the origin. Thus $\widehat{\varphi} \in C^1$ (not C^2, as incorrectly concluded from a true proof *in [7]), and reproduce polynomials up to degree 1.*

For $n = 2$, with our choice of η in $\mathbb{R}^2$ we get $\widehat{\varphi_1}(\omega) = 1 + \mathcal{O}(\|\omega\|^{2(m-1)})$. This yields a smoother $\widehat{\varphi_1} \in C^{2(m-1)-1}(\mathbb{R}^2)$. In particular, φ_1 allows the reproduction of polynomials up to degree $2(m-1)-1$. This is due to the fact that $D^\beta \widehat{\varphi}_1(0) = 0$, if $|\beta| \leq 2(m-1)-1$. Such an approach can also be pursued for polyharmonic B-splines in higher dimensions $n > 2$.

Example 13*(i) The trigonometric polynomial η_1 (19) of the form*

$$\eta_1(\omega_1, \omega_2) = r^2 - \frac{\cos(4\theta)+3}{48} r^4 + R(\omega) r^6 = r^2 + R_3(\omega) r^4$$

is approximately rotation-invariant of order 3. Thus, the corresponding low-pass filter H is at most C^K with $K = \min(\lfloor \alpha + \frac{N}{2} \rfloor, 1)$, because

$$\begin{aligned}\frac{\eta_1(A^T\omega)}{\eta_1(\omega)} &= (a^2+b^2)(1 + r^2 \frac{r^2(R_3(A^T\omega)(a^2+b^2) - R_3(\omega))}{r^2 + R_3(\omega) r^4}) \\ &= (a^2+b^2)(1 + r^2 B(\omega))\end{aligned}$$

with bounded, but discontinuous B.

(ii) The trigonometric polynomial η_2 (20) of the form

$$\eta_2(\omega) = r^2 - \frac{1}{12} r^4 + R_5(\omega) r^6$$

yields

$$\begin{aligned}&\frac{\eta_2(A^T\omega)}{\eta_2(\omega)} \\ &= (a^2+b^2)\left(1 + r^2 \frac{-\frac{1}{12}(a^2+b^2-1)r^2 + (R_5(A^T\omega)(a^2+b^2) - R_5(\omega))r^4}{r^2 - \frac{1}{12}r^4 + R_5(\omega) r^6}\right) \\ &= (a^2+b^2)\left(1 + r^2 B(\omega)\right) = (a^2+b^2)(1 + \mathcal{O}(\|\omega\|^2))\end{aligned}$$

with B continuous at the origin. Therefore, $\frac{\eta_2(A^T\omega)}{\eta_2(\omega)} \in C^2$.

(iii) For the multipliers in Table 2 of the form $\eta(\omega) = r^2 + R_{2m-1}(\omega) r^{2m}$, we get

$$\frac{\eta(A^T\omega)}{\eta(\omega)} = (a^2+b^2)\left(1 + r^{2(m-1)} \frac{r^2(R_{2m-1}(A^T\omega)(a^2+b^2) - R_{2m-1}(\omega))}{r^2 + R_{2m-1}(\omega) r^{2m}}\right).$$

Thus $\frac{\eta(A^T\omega)}{\eta(\omega)} \in C^{2m-3}$.

For the construction and orthonormalization of the corresponding wavelet functions, which span the space W_j with $V_j \oplus W_j = V_{j+1}$, we have to consider the autocorrelation filter

$$M(\omega) = \sum_{k\in\mathbb{Z}^2} |\varphi(\omega + 2\pi k)|^2.$$

In the same way as for H it is interesting, how smooth M is with respect to the choice of η.

Theorem 14 *Let ν, η be as in Theorem 12. If $\eta(\omega) = r^2 + \sum_{k=2}^{m} c_k r^{2k} + R_{2m+1}(\omega) r^{2(m+1)}$ with $c_k \neq 0$ for all k, then M is at most continuous. I.e. $M \in C^{K_3}$ with $K_3 = \min\{\lfloor \alpha + \frac{N}{2} \rfloor, 1\} - 1$.*

If η is of the particular form $\eta(\omega) = r^2 + R_{2m+1}(\omega) r^{2(m+1)}$, then M is C^{K_4}, $K_4 = \min\{\lfloor \alpha + \frac{N}{2} \rfloor, 2m - 1\} - 1$.

For the case $\alpha + \frac{N}{2} \in \mathbb{N}$, the first term can be omitted from both minima.

PROOF. We prove the theorem by considering the decay of the Fourier coefficients of M. We have $M \in L^1([0, 2\pi]^2)$, since by Lebesgue's dominated convergence theorem

$$\begin{aligned} \frac{1}{(2\pi)^2} \int\limits_{[0,2\pi]^2} |M(\omega)|\, d\omega \quad &= \quad \frac{1}{(2\pi)^2} \int\limits_{[0,2\pi]^2} \sum_{k\in\mathbb{Z}^2} |\widehat{\varphi}(\omega + 2\pi k)|^2\, d\omega \\ = \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2} |\varphi(x)|^2\, d\omega < \infty, & \end{aligned}$$

if $2\alpha + N > 1$, which is always true in our setting $N \geq 1$, $\alpha > 0$.

The Fourier coefficients of M by the same arguments are

$$\widehat{M}(l) = \frac{1}{(2\pi)^2} \int\limits_{[0,2\pi]^2} \sum_{k\in\mathbb{Z}^2} |\widehat{\varphi}(\omega + 2\pi k)|^2 e^{-i\langle\omega,l\rangle}\, d\omega, \quad l \in \mathbb{Z}^2.$$

We consider two cases for the function

$$|\widehat{\varphi}(\omega)|^2 = \left(\frac{\eta(\omega)^2}{\|\omega\|^4}\right)^{\alpha+\frac{N}{2}}.$$

Due to the period zeros of the numerator we deduce that $|\widehat{\varphi}|^2$ is at most $C^{\lfloor \alpha+\frac{N}{2}\rfloor}$, if $\alpha+\frac{N}{2}\notin\mathbb{N}$.

1. Case: η is of the general form (23). Then

$$
\begin{aligned}
\eta(\omega)^2 &= (r^2+\sum_{k=2}^{m} c_k r^{2k} + R_{2m+1}(\omega) r^{2(m+1)})^2 \\
&= r^4 + \sum_{k=2}^{m} d_k r^{2(k+1)} + \widetilde{R}_{2m+3}(\omega) r^{2m+4}
\end{aligned}
$$

in a neighborhood of the origin. Here, d_k are certain coefficients, $\widetilde{R}_{2m+3}$ is a bounded term and $r=\sqrt{\omega_1^2+\omega_2^2}=\|\omega\|$. Thus

$$
|\widehat{\varphi}(\omega)|^2 = \left(\frac{r^4+\mathcal{O}(r^6)}{r^4}\right)^{\alpha+\frac{N}{2}} = (1+\mathcal{O}(r^2))^{\alpha+\frac{N}{2}},
$$

from which we deduce $|\widehat{\varphi}|^2 \in C^{K_3'}$, $K_3'=\min\{\lfloor\alpha+\frac{N}{2}\rfloor,1\}$.

2. Case: η is of the particular form $\eta(\omega)=r^2+R_{2m+1}(\omega)r^{2(m+1)}$, then $\eta(\omega)^2 = r^4+\widetilde{R}_{2m+3}(\omega)r^{2(m+2)}$ for a bounded term $\widetilde{R}_{2m+3}$. Then

$$
|\widehat{\varphi}(\omega)|^2 = (1+\mathcal{O}(\|\omega\|^{2m}))^{\alpha+\frac{N}{2}} \in C^{K_4'},
$$

with $K_4'=\min\{\lfloor\alpha+\frac{N}{2}\rfloor,2m-1\}$.

In both cases we get for the decay of the Fourier transform

$$
\int_{\mathbb{R}^2} |\widehat{\varphi}(\omega)|^2 e^{-ik\omega}\,d\omega \le \frac{\text{const}}{1+|k|^\gamma}, \quad \gamma\le K_j'+1,\ j=3,4.
$$

Hence, $M\in C^{K_j'-1}$. $\square$

7 Wavelet bases

In this section, we construct Riesz bases spanning the orthogonal complement W_j in $V_{j+1}=W_j\oplus V_j$, i.e., prewavelet bases. It is well known that the minimal number of prewavelets ψ_d, $d=1,\ldots,D$, which span have

$$
W_j = \overline{\operatorname{span}\{2^{j/2}\psi_d(A^j\bullet-k), k\in\mathbb{Z}^2, d=1,\ldots,D\}}^{L^2(\mathbb{R}^2)}
$$

depends on the number of cosets $q=|\det A|$ of $A(\mathbb{Z}^2)$ in $\mathbb{Z}^2$. In fact, $D=q-1$. A detailed proof of this fact can be found in [4]. Corresponding wavelet bases can be constructed in orthonormalizing the prewavelet bases.

In the following, as an example, we consider prewavelet bases in the quincunx case with dilation matrix

$$A = \begin{pmatrix} & 1\ 1 \\ -1\ 1 & \end{pmatrix}. \tag{24}$$

Since $\det A = 2$, only one wavelet ψ spans the space W_{-1}. This case is especially interesting for image processing, since the data is subsampled by a factor of only $\det A = 2$ in each decomposition step, in comparision to 4 in a tensor product approach. However, we note that our approach does not work with other dilation matrix also representing the quincunx lattice, as for example

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix},$$

since they are not scaled rotations. Further details on similarities and differences of multiresolutions based on these dilation matrices may be found e.g. in [8,9].

For arbitrary dilation matrices, a general method to construct the corresponding (pre-)wavelets uses unitary polyphase matrices. For a comprehensive introduction, we refer to [4,10].

7.1 Prewavelets in the quincunx case

There are several possibilities to construct prewavelet bases $\{\psi_{j,k} = 2^{j/2}\psi(A^j \bullet -k), k \in \mathbb{Z}^2\}$ spanning the space W_j for dilations A as in (24). For example, it is well known that

$$\widehat{\psi}(A^T\omega) = e^{-i\omega_1}\overline{H(\omega + (\pi,\pi)^T)}M(\omega + (\pi,\pi)^T)\widehat{\varphi}(\omega) \tag{25}$$

generates a Riesz basis of W_j for the quincunx case. In general, this basis is not orthonormal $\langle\psi_{j,k}, \psi_{j,l}\rangle \neq \delta_{k,l}$. The corresponding dual basis is given by

$$\widehat{\psi}_{\sim}(A^T\omega) = e^{-i\omega_1}\overline{H(\omega + (\pi,\pi)^T)}\frac{M(\omega + (\pi,\pi)^T)}{M(A^T\omega)}\frac{\widehat{\varphi}(\omega)}{M(\omega)},$$

where M is the autocorrelation function (14). Then all $f \in L^2(\mathbb{R}^2)$ can be represented as

$$f = \sum_{j,k} \langle f, \psi_{\sim,j,k} \rangle \psi_{j,k} = \sum_{j,k} \langle f, \psi_{j,k} \rangle \psi_{\sim,j,k}.$$

An orthonormal wavelet basis is generated by

$$\widehat{\psi}_\perp(A^T\omega) = \sqrt{\frac{M(\omega + (\pi,\pi)^T)}{M(A^T\omega)}}\,\widehat{\psi}(\omega).$$

In this case, a corresponding orthonormal basis of V_0 is generated by integer shifts of the scaling function $\widehat{\varphi}_\perp(\omega) = \widehat{\varphi}(\omega)/\sqrt{M(\omega)}$.

In the following we concentrate on the prewavelets (25), since the other wavelet functions mentioned above are variations of the form $\widehat{\psi}_*(A^T\omega) = \widehat{\psi}(A^T\omega)S_*(\omega)$ for some fraction S_* of shifted and scaled autocorrelation functions M.

7.2 Smoothness and decay

In the following we consider prewavelets (25) constructed from the real multipliers (22). These functions have the same decay properties as $\widehat{\varphi}$:

$$\widehat{\psi}(\omega) = \mathcal{O}\left(\frac{1}{\|\omega\|^{2\alpha+N}}\right) \qquad \text{as} \qquad \|\omega\| \to \infty.$$

For $r = \|\omega\| = \sqrt{\omega_1^2 + \omega_2^2}$, we get in a neighborhood of the origin

$$\begin{aligned}
&H(\omega + (\pi,\pi)^T) \quad = \\
&= \frac{1}{2^\alpha(1-i)^N}\left(\frac{\eta(\omega_1-\omega_2,\omega_1+\omega_2)}{\eta(\omega_1+\pi,\omega_2+\pi)}\right)^{\alpha+\frac{N}{2}} \\
&= \frac{1}{2^\alpha(1-i)^N}\left(\frac{2r^2 + \sum_{k=2}^m c_k(2r^2)^k + R_{2m+1}(\omega_1-\omega_2,\omega_1+\omega_2)2^{m+1}r^{2(m+1)}}{\eta(\omega_1+\pi,\omega_2+\pi)}\right)^{\alpha+\frac{N}{2}} \\
&= \frac{2^{\alpha+\frac{N}{2}}}{2^\alpha(1-i)^N}\left(\frac{2r^2 + \sum_{k=2}^m c_k(2r^2)^k + R_{2m+1}(\omega_1-\omega_2,\omega_1+\omega_2)2^{m+1}r^{2(m+1)}}{\eta(\omega_1+\pi,\omega_2+\pi)}\right)^{\alpha+\frac{N}{2}} \\
&= e^{i\frac{\pi}{4}N}\left(r^2 + \sum_{k=2}^m c_k 2^{k-1} r^{2k} + R_{2m+1}(\omega_1-\omega_2,\omega_1+\omega_2)2^m r^{2(m+1)}\right)^{\alpha+\frac{N}{2}} \\
&\cdot\ \eta(\omega_1+\pi,\omega_2+\pi)^{-(\alpha+\frac{N}{2})}.
\end{aligned} \tag{26}$$

Therefore, the Fourier transform of the wavelet $\widehat{\psi}$ in a neighborhood of the origin behaves as

$$\begin{aligned}
\widehat{\psi}(A^T\omega) &= e^{-i\omega_1}\overline{H(\omega+(\pi,\pi)^T)}M(\omega+(\pi,\pi)^T)\widehat{\varphi}(\omega) \\
&= e^{-i\frac{\pi}{4}N}\left(r^2+\sum_{k=2}^{m}c_k 2^{k-1}r^{2k}+R_{2m+1}(\omega_1-\omega_2,\omega_1+\omega_2)2^m r^{2(m+1)}\right)^{\alpha+\frac{N}{2}} \\
&\quad \cdot\ \eta(\pi,\pi)^{-(\alpha+\frac{N}{2})}M(\omega+(\pi,\pi)^T)\widehat{\varphi}(\omega).
\end{aligned}$$

A closer look at the scaling function $\widehat{\varphi}$ and its behaviour at the origin yields

$$\begin{aligned}
\widehat{\varphi}(\omega) &= \frac{\left(r^2+\sum_{k=2}^{m}c_k r^{2k}+R_{2m+1}(\omega)r^{2(m+1)}\right)^{\alpha+\frac{N}{2}}}{(\omega_1^2+\omega_2^2)^{\alpha}(\omega_1-i\omega_2)^N} \\
&= (\omega_1+i\omega_2)^N\frac{\left(r^2+\sum_{k=2}^{m}c_k r^{2k}+R_{2m+1}(\omega)r^{2(m+1)}\right)^{\alpha+\frac{N}{2}}}{r^{2\alpha}r^{2N}} \\
&= \frac{(\omega_1+i\omega_2)^N}{r^N}\left(\frac{r^2+\sum_{k=2}^{m}c_k r^{2k}+R_{2m+1}(\omega)r^{2(m+1)}}{r^2}\right)^{\alpha+\frac{N}{2}} \\
&= e^{iN\arg\omega}\left(1+\sum_{k=2}^{m}c_k r^{2(k-1)}+R_{2m+1}(\omega)r^{2m}\right)^{\alpha+\frac{N}{2}},
\end{aligned}$$

where $\arg\omega$ denotes the angle of the vector ω with the ω_1-axis. Thus

$$\begin{aligned}
\widehat{\psi}(A^T\omega) &= e^{-i\frac{\pi}{4}N}\left(r^2+\sum_{k=2}^{m}c_k 2^{k-1}r^{2k}+R_{2m+1}(\omega_1-\omega_2,\omega_1+\omega_2)2^m r^{2(m+1)}\right)^{\alpha+\frac{N}{2}} \\
&\quad \cdot\ \eta(\pi,\pi)^{-(\alpha+\frac{N}{2})}M(\pi,\pi)e^{iN\arg\omega}\left(1+\sum_{k=2}^{m}c_k r^{2(k-1)}+R_{2m+1}(\omega)r^{2m}\right)^{\alpha+\frac{N}{2}} \\
&= e^{-i\frac{\pi}{4}N}e^{iN\arg\omega}(r^2+\mathcal{O}(r^4))^{\alpha+\frac{N}{2}}\cdot\eta(\pi,\pi)^{-(\alpha+\frac{N}{2})}M(\pi,\pi)
\end{aligned} \tag{27}$$

for $\|\omega\|\to 0$. If, in particular, $\eta(\omega)=r^2+R_{2m+1}(\omega)r^{2(m+1)}$, then

$$\begin{aligned}
\widehat{\psi}(A^T\omega) &= e^{-i\frac{\pi}{4}N}\left(r^2+R_{2m+1}(\omega_1-\omega_2,\omega_1+\omega_2)2^m r^{2(m+1)}\right)^{\alpha+\frac{N}{2}} \\
&\quad \cdot\ \eta(\pi,\pi)^{-(\alpha+\frac{N}{2})}M(\pi,\pi)e^{iN\arg\omega}(1+R_{2m+1}(\omega)r^{2m})^{\alpha+\frac{N}{2}} \\
&= e^{-i\frac{\pi}{4}N}e^{iN\arg\omega}(r^2+\mathcal{O}(r^{2m}))^{\alpha+\frac{N}{2}}\eta(\pi,\pi)^{-(\alpha+\frac{N}{2})}M(\pi,\pi).
\end{aligned}$$

Hence, $\widehat{\psi}$ is continuous at the origin, and there shows rotation-covariant behaviour. To see this, let R_θ be a rotation operator of the form (1).

$$\widehat{\psi}(A^T R_\theta \omega) = e^{-i\frac{\pi}{4}N} e^{iN(\theta+\arg\omega)}(r^2+\mathcal{O}(r^4))^{\alpha+\frac{N}{2}}\eta(\pi,\pi)^{-(\alpha+\frac{N}{2})}M(\pi,\pi) \sim e^{iN\theta}\widehat{\psi}(A^T\omega).$$

We sum up:

Theorem 15 *Let η be as in Theorem 12. Then the following is true:*

(i) $\psi \in W_2^s(\mathbb{R}^2)$ *with* $s < 2\alpha + N - 1$.

(ii) $\widehat{\psi} \in L^1(\mathbb{R}^2)$ *for* $\alpha + \frac{N}{2} > 1$.

(iii) If $\alpha+\frac{N}{2} > 1$, *then* ψ *is uniformly continuous, bounded, and* $\lim_{\|x\|\to\infty}\psi(x) = 0$.

(iv) Let R_θ *be the rotation operator of form (1). Then* $\widehat{\psi}(A^T R_\theta\omega) \sim e^{iN\theta}\widehat{\psi}(A^T\omega)$ *in a neighborhood of the origin.*

(v) $\omega^\beta\widehat{\psi}(\omega) \in L^1(\mathbb{R}^2)$ *for* $|\beta| < 2\alpha + N - 2$. *Thus, in these cases,* $D^\beta\psi$ *is bounded, uniformly continuous, and vanishes at infinity.*

(vi) $\widehat{\psi}$ *has a zero of order* $2\alpha + N$ *at the origin and has regularity* $\widehat{\psi} \in C^K$ *with* $K = \min\{K_1, K_3\}$ *if* $\eta(\omega)$ *as in (23) with* $c_k \neq 0$ *for all* k, *or* $K = \min\{K_2, K_4\}$ *for* $\eta(\omega) = r^2 + \mathcal{O}(r^{2(m+1)})$.

(vii) In time domain, $\psi(x) = \mathcal{O}(\frac{1}{\|x\|^\gamma})$ *for* $\|x\| \to \infty$, $\gamma \leq K + 1$ *and* $\alpha + \frac{N}{2} > 1$.

PROOF. Claims (i) and (ii) are direct consequences of the decay (3) of $\widehat{\varphi}$, which is handed down to the wavelet $\widehat{\psi}$, and the continuity of $\widehat{\psi}$. (iii) follows from (ii) by properties of the Fourier transform. (iv) is proved above. (v) follows directly from the properties of the Fourier transform.

(vi) $\widehat{\psi}(\omega) = \mathcal{O}(\|\omega\|^{2\alpha+N})$ for $\|\omega\| \to 0$. Thus $\widehat{\psi}$ is $(2\alpha + N) - 1$ times differentiable at the origin. For the general estimate of the regularity we consider the representation

$$\widehat{\psi}(A^T\omega) = e^{-i\omega_1}\overline{H(\omega + (\pi,\pi)^T)}M(\omega + (\pi,\pi)^T)\widehat{\varphi}(\omega).$$

Since translations and dilations are C^∞ transforms, Theorems 12 on H and 14 on M yield the regularity $\widehat{\psi} \in C^{\min\{K_1,K_3\}}$ if $\eta(\omega)$ as in 23 with $c_k \neq 0$ for all k, resp. $\widehat{\psi} \in C^{\min\{K_2,K_4\}}$ for $\eta(\omega) = r^2 + \mathcal{O}(r^{2(m+1)})$.

(vii) Now let $K = \min\{K_1, K_3\}$ or $K = \min\{K_2, K_4\}$ respectively. Then $D^\beta\widehat{\psi} \in L^1(\mathbb{R}^2)$, if $\alpha + \frac{N}{2} > 1$ and $|\beta| \leq K + 1$. In this case, $x^\beta\psi$ is bounded, uniformly continuous, and vanishes at infinity. Consequently, $\psi(x) = \mathcal{O}(\frac{1}{\|x\|^{|\beta|}})$ for $\|x\| \to \infty$ and the claim is proved. □

To identify the wavelets with certain differential operators, we write $\widehat{\psi}$ in a different form, starting out from (27).

$$\begin{aligned}\widehat{\psi}(A^T\omega) &= e^{-i\frac{\pi}{4}N}e^{iN\arg\omega}(r^2+\mathcal{O}(r^4))^{\alpha+\frac{N}{2}}\eta(\pi,\pi)^{-(\alpha+\frac{N}{2})}\\ &= e^{-i\frac{\pi}{4}N}e^{iN\arg\omega}((\omega_1^2+\omega_2^2)^{\alpha+\frac{N}{2}}+\mathcal{O}(r^{\alpha+2N+2}))\eta(\pi,\pi)^{-(\alpha+\frac{N}{2})}\\ &= e^{-i\frac{\pi}{4}N}(\omega_1^2+\omega_2^2)^{\alpha}(\sqrt{\omega_1^2+\omega_2^2}e^{i\arg\omega})^N\frac{M(\pi,\pi)}{\eta(\pi,\pi)^{\alpha+\frac{N}{2}}}+\mathcal{O}(r^{4\alpha+2N+2})\\ &= e^{-i\frac{\pi}{4}N}(\omega_1^2+\omega_2^2)^{\alpha}(\omega_1+i\omega_2)^N\eta(\pi,\pi)^{-(\alpha+\frac{N}{2})}+\mathcal{O}(r^{4\alpha+2N+2}).\end{aligned}$$

This implies

$$\widehat{\psi}(\omega)=\frac{\sqrt{2}^N}{2^\alpha}(\omega_1^2+\omega_2^2)^{\alpha}(\omega_1+i\omega_2)^N\eta(\pi,\pi)^{-(\alpha+\frac{N}{2})}+\mathcal{O}(r^{4\alpha+2N+2}).$$

Therefore, the wavelet transform of a test function $f\in\mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned}\langle f,\psi(\bullet-k)\rangle &= \frac{1}{(2\pi)^2}\langle\widehat{f},(\psi(\bullet-k))\widehat{}\,\rangle\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{f}(\omega)e^{i\langle\omega,k\rangle}\overline{\widehat{\psi}(\omega)}\,d\omega\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{f}(\omega)(\omega_1^2+\omega_2^2)^{\alpha}(\omega_1-i\omega_2)^N\overline{\widehat{\Phi}(\omega)}e^{i\langle\omega,k\rangle}\,d\omega\\ &= (-\Delta)^\alpha\left(-i\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)^N\{f*\overline{\Phi}\}(k)\end{aligned}$$

behaves as a Laplacian of order α modified by a differential operator of Wirtinger type of order N. Here,

$$\begin{aligned}\widehat{\Phi}(\omega) &= \frac{\widehat{\psi}(\omega)}{(\omega_1^2+\omega_2^2)^{\alpha}(\omega_1+i\omega_2)^N}\\ &= e^{-\frac{i}{2}(\omega_1+\omega_2)}(1+\mathcal{O}(\|\omega\|))\widehat{\varphi}(A^{-T}\omega)\quad\text{for }\|\omega\|\to 0\end{aligned}$$

is a lowpass smoothing kernel with approximate rotation-covariant behaviour in a neighborhood of the origin. The wavelet thus can be represented as

$$\psi(x_1,x_2)=(-\Delta)^\alpha\left(-i\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)^N\overline{\Phi(x_1,x_2)},$$

i.e., the derivative of order $2\alpha + N$ of the smoothing kernel $\overline{\Phi} \in L^2(\mathbb{R}^2)$ in the sense of distributions.

8 Implementations aspects

The coefficients of the refinement filter H depend on the choice of the $(2\pi, 2\pi)$–periodic localizing function ν. Moreover, for non-integer α, the filter H has in general an infinite impulse response and is non-causal. Therefore, a classical implementation by convolution in space domain would tend to be rather costly and would lead to accuracy problems because of the necessity to truncate the filters.

Since the filter H is given explicitly in Fourier domain (13), it is more convenient to implement the corresponding wavelet transform in the Fourier domain using FFTs. We again consider the quincunx case, where we have to take into account just one wavelet filter. The discrete Fourier transform of the image data is filtered by multiplying with the refinement and the wavelet filter. Exploiting symmetries, we can downsample the data by a factor of 2 and use an inverse discrete Fourier transform on the reduced data. This yields a fast and stable algorithm.

As an example, we calculated the image of a wavelet with the fast algorithm (see Fig. 4). We measured for the precalculation of the filters, which is done only once, for a sample size of 128×128 a duration of 68,71 sec. The calculation of the wavelet then needs only 0,09 sec. For the standard test image 'cameraman' (size 256×256) precalculation of the filters needs 543,98 sec. Wavelet analysis and synthesis take 0,71 sec resp. 0,44 sec and yield an l^∞-reconstruction error of $5,12 \cdot 10^{-13}$. For the larger standard test image 'peppers' (size 512×512) precalculation of the filters needs 7028,50 sec. Wavelet analysis and synthesis take 3,17 sec resp. 2,03 sec and yield an l^∞-reconstruction error of $5,03 \cdot 10^{-13}$. Both wavelet analysis and synthesis steps were calculated with the same parameters as in Fig. 4.

9 Conclusion

We presented a new family of complex multiresolution bases in $L^2(\mathbb{R}^2)$. Our approach considered localizations of the rotation-covariant function of the form $\rho(x_1, x_2) = C(\alpha, N)(x_1^2 + x_2^2)^{\alpha-1}(x_1 + ix_2)^N$ (times a logarithmic factor if $\alpha \in \mathbb{N}$). This yields nonseparable complex-valued multiresolution bases of $L^2(\mathbb{R}^2)$ for scaled rotation dilation matrices.

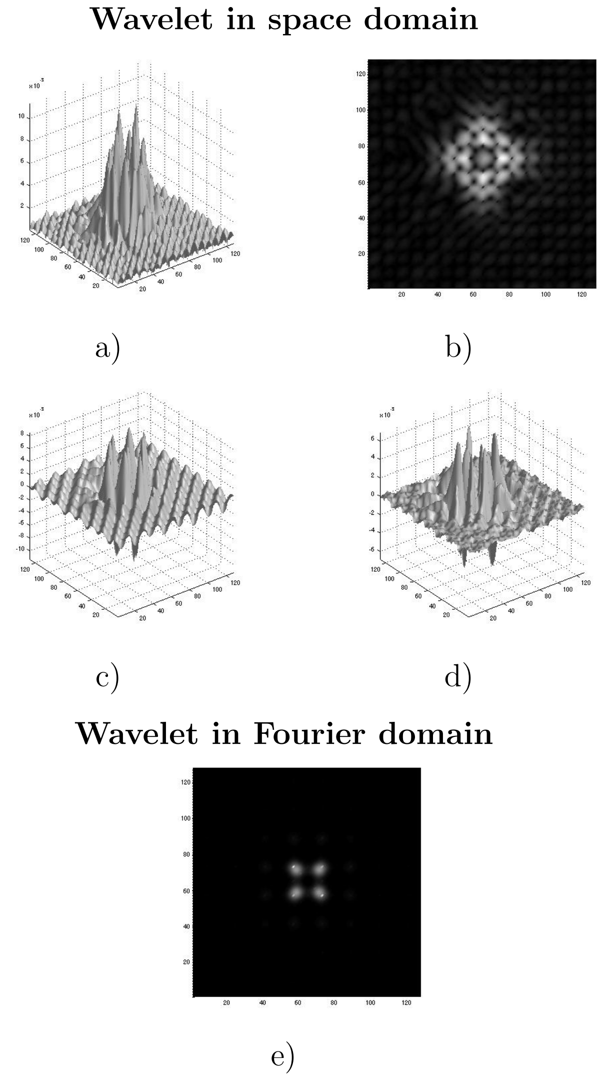


Fig. 4. Wavelet calculated using the fast algorithm for the parameters $\alpha = 1$, $N = 1$ and localizing multiplier ν based on the trigonometric polynomial η in(20). a) Modulus of wavelet function in space domain and b) top view. c) Real part and d) imaginary part of wavelet in space domain. e) Top view on modulus of wavelet in Fourier domain.

The parameter $\alpha \in \mathbb{R}^+$ can be chosen arbitrarily; in particular, non-integer. This gives flexibility: We can control the smoothness and decay of the resulting wavelets. With the integer parameter N, we are able to influence rotation-covariance properties. Both degrees of freedom can be desirable for applications in image analysis.

The smoothness and decay properties of the resulting wavelets depend on the choice of the localizing multiplier. There, we gave a new class of multipliers of the form $\eta(\omega_1, \omega_2) = \omega_1^2 + \omega_2^2 + \mathcal{O}(\|\omega\|^{2m})$, which yield smoother functions and filters. These may also be used to improve the properties of the classical polyharmonic B-splines ($N = 0$).

Since our new family of complex rotation-covariant functions yields multiresolution bases, we can apply them in the DWT algorithm. We thus have a non-redundant complex wavelet transform and perfect reconstruction. The transform leads easily to an efficient filterbank implementation using Mallat's algorithm. The wavelet filters can also be specified to yield various types of decompositions; i.e., orthogonal, semi-orthogonal, or bi-orthogonal. Moreover, a FFT-based implementation can provide an efficient and fast algorithm.

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